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# VARIATIONAL METHOD OF INVESTIGATION OF THREE-DIMENSIONAL MIXED PROBLEMS OF A PLANE CUT IN AN ELASTIC MEDIUM IN THE PRESENCE OF SLIP AND ADHESION OF ITS SURFACES*

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A general three-dimensional static problem concerning an arbitrary crack occupying a plane region in an infinite elastic medium is considered. It is assumed that the development of the crack occurs under the combined action of tenson, compression, and shear loads in relation to the plane of crack, and is accompanied by the formation of regions, where its surfaces come into contact. In unknown beforehand zones of contact there is a friction with the coefficient depending on normal pressure and the magnitude of relative tangential displacement of the surfaces. Regions of local adhesion and slippage may be formed. An equivalent variational statement of the input boundary value problem is given. This problem is formulated in the form of systems of equalities and inequalities as the problem of minimization of uneven non-quadratic functionals dependent on jumps of displacements in the region of the crack. Conditions of existence and uniqueness of solution in the Sobolev — Slobodetskii space of fractional order are indicated. Certain properties of solutions are established and, also, the qualitative results of its integral characteristics dependence on the form of the crack and the parameters of the law of friction. The variational formulation of the problem obtained here allows to construct its numerical solution using methods of mathematical programming.

Problems of cracks with partly contacting surfaces without friction were considered in the axisymmetric case /1/ and in the general cases /2/. Plane problem on contacting surfaces of cracks with friction between them taken into account were studied in /3-6/. Contact problems with conditions of friction similar to those considered below were studied in /7/. Solution of some contact problem with the dependence of the coefficient of friction on pressure was derived in /8/.

**1. Statement of the limit problem.** The equilibrium of elastic space with a plane cut under the action of antisymmetric body and surface forces applied to the surfaces of the cut is considered. The surface of the cut on some (not known beforehand) part of the region taken by it, may contact one another, while on the remaining part they do not touch each other. The stresses on the open part of the cut coincide with specified surface loads; the stresses on each surface in the region of contact represent the sum of external load and the action of the other (contacting) surface. The component normal to the surface of the cut pressure is of constant sign and related to its tangential component by the law of friction with conditions of slippage and adhesion.

In the part of contact, where the relative displacement of the surfaces is absent (their adhesion takes place) the magnitude of the tangential component of interaction does not exceed the product of the friction coefficient and pressure. The coefficient of friction in the region of adhesion depends on pressure at the considered point. In the remaining contact region of the surfaces of the cut (in the region of slippage), where a relative displacement exists, the tangent component of interaction of surfaces reaches the value of the product of the friction coefficient and pressure, and is directed along the relative displacement. The friction coefficient in the region of slippage depends on pressure and the magnitude of relative displacement of surfaces at the considered point. The boundary between regions of slippage and adhesion must be determined in the course of solving the problem.

Let the cut occupy the region $\Omega$ with boundary $\Gamma$ in the plane $Z=0$. We denote by $\mathbf{Q}(x, y, z)$ and $\mathbf{R}^{\pm}(x, y)$ $(\mathbf{R}^- = -\mathbf{R}^+ = r)$ the densities of the volume and surface loads. For the normal $w$ and tangent $\mathbf{u}_\tau$ of the components of displacement jump (taken from the lower surface), and for the normal $\sigma_z$ and tangent $\boldsymbol{\sigma}_\tau$ surface stresses we, then, have the following conditions in the plane $Z=0$:

throughout the whole region of the cut

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$$\sigma_z \leqslant r_z, \quad w = 0; \quad \sigma_z = r_z, \quad \boldsymbol{\sigma}_\tau = \mathbf{r}_\tau, \; w < 0 \tag{1.1}$$

in the region of contact $E$, where $w = 0$

$$|\boldsymbol{\sigma}_\tau - \mathbf{r}_\tau| \leqslant f(p)\, p, \quad |\mathbf{u}_\tau| = 0 \tag{1.2}$$

$$\boldsymbol{\sigma}_\tau - \mathbf{r}_\tau = -[f(p) + g(p, |\mathbf{u}_\tau|)]\, p\, \mathbf{u}_\tau / |\mathbf{u}_\tau|, \qquad |\mathbf{u}_\tau| \neq 0 \tag{1.3}$$

and outside region $\Omega$

$$w = |\mathbf{u}_\tau| = 0 \tag{1.4}$$

Here $p = -(\sigma_z - r_z)$ is the pressure and $f(p)$, $g(p, |\mathbf{u}_\tau|)$ are functions that specify the dependence of the friction coefficient on pressure and relative slippage of the surfaces.

In formulating the problem with the process of loading of the space with the cut taken into account it is necessary, similarly to what was done in /9/ for contact problems, to substitute the increment of $\mathbf{u}_\tau$ corresponding to the increment of the loading parameter, on which depend the external loads $\mathbf{Q}$ and $\mathbf{r}$, for the function $\mathbf{u}_\tau$.

The unknown stress and displacement fields can be represented in the form of two fields, the first of which corresponds to the state of infinite space without a cut subjected to the action of volume forces $\mathbf{Q}$, and the second to the state of the space with the cut $\Omega$ with antisymmetric surface loads $(\mathbf{R}^\pm \pm \boldsymbol{\sigma}_0)$. Here $\boldsymbol{\sigma}_0(\sigma_z^\circ, \boldsymbol{\sigma}_\tau^\circ)$ are stresses at points of the plane $Z = 0$ of the continuous space under the action of forces $\mathbf{Q}$. Subsequently we shall define the fields appertaining to the second state taking stresses $\boldsymbol{\sigma}_0$ as known. For the unknown fields we retain the same notation $\sigma_z, \boldsymbol{\sigma}_\tau, w, \mathbf{u}_\tau$.

The projections of the displacement jump at points of the plane cut in the infinite medium (in the absence of volume loads) and the stresses at which the jump is realized are related by the known formulas /10/

$$\sigma_z = \frac{G}{2(1-\nu)} F^{-1}[|\xi| F(w)], \quad \boldsymbol{\sigma}_\tau = \frac{G}{2(1-\nu)} F^{-1}[A(\xi) F(\mathbf{u}_\tau)] \tag{1.5}$$

$$A(\xi) = |\xi| \begin{Vmatrix} 1 - \nu\eta_2^2 & \nu\eta_1\eta_2 \\ \nu\eta_1\eta_2 & 1 - \nu\eta_1^2 \end{Vmatrix}, \quad \eta_i = \frac{\xi_i}{|\xi|}$$

where $F$ and $F^{-1}$ are operators of the direct and inverse Fourier transformations with parameter $\xi(\xi_1, \xi_2)$, $G$ is the shear modulus, and $\nu$ is the Poisson coefficient of the medium.

It is seen from the first of equalities (1.5) that the normal stresses at points of region $\Omega$ are independent of the jump of tangent displacement. This allows us to separate the input problem (1.1) – (1.4) into two, which are solved successively. In the first problem the region of contact of surfaces $E$ and the normal stresses at points of $\Omega$ are determined from conditions

$$\boldsymbol{\sigma}_\tau = 0; \quad \sigma_z(w) \leqslant r_z - \sigma_z^\circ, \quad w = 0; \quad \sigma_z = r_z - \sigma_z^\circ, \quad w < 0 \tag{1.6}$$

In the second, for known pressure in region $E$ we obtain $\boldsymbol{\sigma}_\tau$ and $\mathbf{u}_\tau$ (as well as regions of slippage and adhesion within the boundaries of $E$) from conditions

$$|\boldsymbol{\sigma}_\tau - \mathbf{r}_\tau + \boldsymbol{\sigma}_\tau^\circ| \leqslant f(p)\, p, \quad |\mathbf{u}_\tau| = 0 \tag{1.7}$$

$$\boldsymbol{\sigma}_\tau - \mathbf{r}_\tau + \boldsymbol{\sigma}_\tau^\circ = -[f(p) + g(p, |\mathbf{u}_\tau|)]\, p\mathbf{u}_\tau / |\mathbf{u}_\tau|, \quad |\mathbf{u}_\tau| > 0 \text{ in } E$$

$$\boldsymbol{\sigma}_\tau = \mathbf{r}_\tau - \boldsymbol{\sigma}_\tau^\circ \text{ in } \Omega / E \tag{1.8}$$

The representation of unknown fields in the form of a sum of two fields used here is valid in spite of nonlinearity of input boundary conditions (1.1) – (1.3). Having found $\sigma_z, \boldsymbol{\sigma}_\tau$ from the solution of problem (1.6), (1.7) and adding to them $\sigma_z^\circ$, $\boldsymbol{\sigma}_\tau^\circ$, we obtain functions that satisfy (1.1) – (1.3), since the passing to the auxilliary problem (1.6), (1.7) does not change the displacement jumps (in the continuous space under the action of forces $\mathbf{Q}$ they are absent) and pressures (they represent the difference between the true and external normal stresses; in the auxilliary problem both these fields are altered in comparison with the input problem by the quantity $-\sigma_z^\circ$).

**2. The variational problem of determination of pressure and the region of contact.** Consider problem (1.6). We shall show that it leads to the equivalent variational problem of minimizing the functional of potential energy of the set of kinematically admissible normal components of the displacement jump.

**Theorem 2.1.** The problem (1.6) with $(r_z - \sigma_z^\circ) \in H_{-1/2}(\Omega)$ is equivalent to the variational problem

$$\min_{w \in V} \left\{ F_1(w) = \int_\Omega \left[ \frac{1}{2} \sigma_z(w) w - (r_z - \sigma_z^\circ) w \right] dx\, dy \right\} \tag{2.1}$$

$$V: w \leqslant 0, \quad w \in \overset{\circ}{H}_{1/2}(\Omega)$$

where $\sigma_z(w)$ is an operator specified by the first of formula (1.5) and $H_s$, $H_s^\circ$ are the Sobolev – Slobodetskii spaces /11/.

The proof is constructed by the scheme developed below for a more complicated case of determining tangential displacement jumps $\mathbf{u}_\tau$.

In solving numerically the variational problem (2.1), a high effectiveness (*) was shown of the method which uses in the transition from a continuous problem to a discrete one bilinear splines and, then, the minimization by the method of projecting the gradient with automatic selection of the step /12/.

**3. The variation of problem for determining shear stresses and displacements**. Let us consider now the problem (1.7), (1.8) of finding $\mathbf{u}_\tau(u_x, u_y)$. We shall show that it is equivalent to the variational problem of minimization of the functional of potential energy (taking into account the works of friction forces) on the tangential components of the displacement jump $\mathbf{u}_\tau$.

**Theorem 3.1.** Let

$$[pg(p, |\mathbf{u}_\tau|) + pf(p)] \geqslant 0 \in H_{-1/2}(\Omega), \quad (\boldsymbol{\sigma}_\tau^\circ - \mathbf{r}_\tau) \in H_{-1/2}(\Omega)$$

$$G(p, |\mathbf{u}_\tau|) = \int_0^{|\mathbf{u}_\tau|} g(p, \xi)\, d\xi \in \overset{\circ}{H}_{1/2}(\Omega)$$

where $g(x, y)$ is a continuous function of argument $y$ and $g_y'(x, y) \geqslant 0$. Then problem (1.7), (1.8) for $p$ known from the solution of (2.1) is equivalent to the following variational problem:

$$\min_{\mathbf{u}_\tau \in \overset{\circ}{H}_{1/2}(\Omega)} \left\{ F_2(\mathbf{u}_\tau) = \int_\Omega \left[ \frac{1}{2} \boldsymbol{\sigma}_\tau(\mathbf{u}_\tau) \mathbf{u}_\tau - \mathbf{r}_\tau \mathbf{u}_\tau + \boldsymbol{\sigma}_\tau^\circ \mathbf{u}_\tau + p(f(p)|\mathbf{u}_\tau| + G(p, |\mathbf{u}_\tau|)) \right] dx\, dy \right\} \tag{3.1}$$

The proof consists of passing to a variational inequality which reprsents some form of conditions of minimum of $F_2(\mathbf{u}_\tau)$ and then from the establishment of equivalence of that inequality to the input boundary value problem.

Let us prove at the beginning that the function $\mathbf{u}_\tau^\circ$ minimizes the functional $F_2(\mathbf{u}_\tau)$ if and only if it $\forall \mathbf{u}_\tau \in H_{1/2}^\circ(\Omega)$ satifies the variational inequality ($\boldsymbol{\sigma}_\tau(\mathbf{u}_\tau^\circ)$ is defined by the second of equalities (1.5))

$$\int_\Omega \{\mathbf{a}(\mathbf{u}_\tau - \mathbf{u}_\tau^\circ) + p[f(p) + g(p, |\mathbf{u}_\tau^\circ|)](|\mathbf{u}_\tau| - |\mathbf{u}_\tau^\circ|)\}\, dx\, dy = \tag{3.2}$$

$$l(\mathbf{u}_\tau, \mathbf{u}_\tau^\circ) \geqslant 0$$

$$\mathbf{a} = \boldsymbol{\sigma}_\tau(\mathbf{u}_\tau^\circ) - \mathbf{r}_\tau + \boldsymbol{\sigma}_\tau^\circ$$

Let us consider first the form

$$W(\mathbf{u}_\tau, \mathbf{u}_\tau) = \int_\Omega \boldsymbol{\sigma}_\tau(\mathbf{u}_\tau) \mathbf{u}_\tau\, dx\, dy$$

which defines the quadratic part of the functional $F_2$.

By virtue of Parseval equality and the second of formulas (1.5) we have

$$W = \int_{R^2} F(\mathbf{u}_\tau) A(\xi) \bar{F}(\mathbf{u}_\tau)\, d\xi_1\, d\xi_2 \geqslant \tag{3.3}$$

$$\frac{G(1 - |\nu|)}{4\pi^2(1 - \nu)} \int_{R^2} |\xi| [|F(u_x)|^2 + |F(u_y)|^2]\, d\xi_1\, d\xi_2 \geqslant 0$$

The increment of functional $F_2$ at point $\mathbf{u}_\tau^\circ$ is of the form

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*) Gol'dshtein R.V. and Zazovskii A.F., Numerical method of solving three-dimensional problems of the theory of elasticity for bodies with plane cuts with allowance for contact of their surfaces. Moscow, Institute of Problems of Mechanics Akad. Nauk SSSR, Preprint No.192, 1982.

$$\Delta F_2(\mathbf{u}_\tau^\circ) = l(\mathbf{u}_\tau, \mathbf{u}_\tau^\circ) + \frac{1}{2} W(\mathbf{u}_\tau - \mathbf{u}_\tau^\circ, \mathbf{u}_\tau - \mathbf{u}_\tau^\circ) + \tag{3.4}$$
$$\int_\Omega \{G(p, |\mathbf{u}_\tau|) - G(p, |\mathbf{u}_\tau^\circ|) - g(p, |\mathbf{u}_\tau^\circ|)[|\mathbf{u}_\tau| - |\mathbf{u}_\tau^\circ|]\}\, dx\, dy$$

Let now the inequality (3.2) be satisfied, which guarantees that the first term in (3.4) is nonnegative. The second term is also nonnegative by virtue of (3.3). The continuity of the function $g$ with respect to $y$ is ensured by the presence of the derivative $G_y'(x, y)$, and $G_y'(x, y) \equiv g(x, y)$; the condition $g_y'(x, y) \geqslant 0$ ensures the convexity downward with respect to $y$ of the function $G(x, y)$. From this follows the nonnegative character of $\forall \mathbf{u}_\tau$ of the sum of the last terms in the right-hand side of (3.4). Consequently the fulfillment of (3.2) under the conditions of the theorem, ensures the inequality $\Delta F_2(\mathbf{u}_\tau^\circ) \geqslant 0$, i.e. the condition of minimum of $F_2$ when $\mathbf{u}_\tau = \mathbf{u}_\tau^\circ$.

Let on the contrary it be known that for $\mathbf{u}_\tau^\circ$ the minimum of $F_2(\mathbf{u}_\tau)$ obtains. Suppose that with this for some $\mathbf{u}_\tau^*$ an inequality inverse to (3.2) is fulfilled. Let us consider $\mathbf{u}_\tau = \mathbf{u}_\tau^\circ + \lambda(\mathbf{u}_\tau^* - \mathbf{u}_\tau^\circ)$, $\lambda \geqslant 0$. The first term in (3.4) which coincides with the left-hand part of (3.2) is proportional to $\lambda$ the second is proportional to $\lambda^2$, and the remaining group of terms in (3.4) is $o(\lambda)$ by virtue of differentiability of the function $G(p, |\mathbf{u}_\tau|)$, with respect to $|\mathbf{u}_\tau|$. In this way the sign of $\Delta F_2$ is determined by terms linear in $\lambda$, and the nonfulfillment of (3.2) leads to the violation of the condition of minimum when $\mathbf{u}_\tau = \mathbf{u}_\tau^\circ$. The equivlance of (3.1) and (3.2) is established.

We shall now show the equivalence of the solution of variational inequality to the solution of the input boundary value problem. Let initially conditions (1.7), (1.8) be satisfied for the function $\mathbf{u}_\tau^\circ$. Then the left-hand side of inequality (3.2) assumes the form

$$\int_\Omega \{\mathbf{a}\mathbf{u}_\tau + p[f(p) + g(p, |\mathbf{u}_\tau^\circ|)]\,|\mathbf{u}_\tau|\}\, dx\, dy = K(\mathbf{u}_\tau, \mathbf{u}_\tau^\circ) \tag{3.5}$$

Outside the region of contact the integrand of (3.5) is zero by virtue of condition (1.8) and identity $p \equiv 0$, while in the region of contact this expression is nonnegative by virtue of conditions (1.7). Thus for the function $\mathbf{u}_\tau^\circ$ which satisfies conditions (1.7) and (1.8) the variational inequality (3.2) is satisfied.

Let now the inequality (3.2) be satisfied. We introduce for the function $\mathbf{u}_\tau^\circ$ the boundary conditions (1.7) and (1.8). We reduce (3.2) to the form

$$K(\mathbf{u}_\tau, \mathbf{u}_\tau^\circ) \geqslant K(\mathbf{u}_\tau^\circ, \mathbf{u}_\tau^\circ) \tag{3.6}$$

It was shown in /13/ that for any $\mathbf{m} \in H_{1/2}^\circ(\Omega)$, $n \geqslant 0 \in H_{-1/2}(\Omega)$ the representation

$$\int_\Omega n\,|\mathbf{m}|\, dx\, dy = \sup_{\alpha \in H_{-1/2}(\Omega),\ |\alpha| \leqslant n} \left[\int_\Omega \alpha\mathbf{m}\, dx\, dy\right]$$

is valid. It is now possible to bring (3.6) to the form

$$K(\mathbf{u}_\tau, \mathbf{u}_\tau) = \int_\Omega \beta_0 \mathbf{u}_\tau\, dx\, dy = \sup_{\beta \in B} \int_\Omega \beta\mathbf{u}_\tau\, dx\, dy \geqslant K(\mathbf{u}_\tau^\circ, \mathbf{u}_\tau^\circ) \tag{3.7}$$
$$\beta_0 \in B, \quad B\colon \beta \in H_{-1/2}(\Omega), \quad |\beta - \mathbf{a}| \leqslant p[f + g]$$

The functional $K(\mathbf{u}_\tau, \mathbf{u}_\tau)$ is linear with respect to $\mathbf{u}_\tau$ and is specified throughout the space $H_{1/2}^\circ(\Omega)$. The inequality (3.7) indicates its boundedness from below, which is only possible when

$$|\beta| = 0, \quad |\mathbf{a}| \leqslant p(f + g) \tag{3.8}$$

Let us consider again the variational inequality (3.6), and set in it $|\mathbf{u}_\tau| \equiv 0$. We then have

$$\int_\Omega [\mathbf{a}\mathbf{u}_\tau^\circ + p(f + g)\,|\mathbf{u}_\tau^\circ|]\, dx\, dy \leqslant 0 \tag{3.9}$$

which with (3.8) taken into account is only possible when the integrand in expression (3.9) is zero.

The combination of this condition with (3.8) is equivalent to the input boundary conditions (1.7) and (1.8). The Theorem 3.1. is proved.

The Theorems 2.1 and 3.1 enable us to reduce the investigations of existence and uniqueness of solution of boundary value problems (1.6), (1.7), and (1.8) to the determination of conditions of existence and uniqueness of the minimum of functionals $F_1$ and $F_2$. The functions which minimize $F_1$ and $F_2$ must belong to those classes in which the equivalence of the boundary value and variational problems was established.

For the presence of a unique minimum in $H_{1/2}^\circ(\Omega)$ of functionals $F_1$ and $F_2$ it is sufficient that their continuity, strict convexity, the convexity of the set of admissible functions and, also, the conditions

$$F_1 \to \infty, \quad \| w \| \to \infty; \quad F_2 \to \infty, \quad | \mathbf{u}_\tau | \to \infty \tag{3.10}$$

are satisfied /13/.

**Theorem 3.2.** If $(r_z - \sigma_z^\circ) \in H_{-1/2}(\Omega)$, then the functional $F_1$ has a unique minimum $w \in H_{1/2}^\circ(\Omega)$.

**Theorem 3.3.** Let all conditions of Theorem 3.1 of the equivalence of the boundary value and the variational problems of determining $\mathbf{u}_\tau$ be satisfied. Let moreover $f(p) \geqslant 0$ and $g(p, |\mathbf{u}_\tau|)$ be a bounded function. Then the functional $F_2$ has a unique minimum $\mathbf{u}_\tau \in H_{1/2}^\circ(\Omega)$.

The proof of these theorems is reduced to the verification of properties of functionals $F_1$ and $F_2$ indicated above. They are accomplished, since $F_1$ is the sum of bounded coercivity quadratic and bounded linear functionals, and functional $F_2$ can be represented in the form of such sum with the addition of a continuous convex functional. The continuity of respective functionals is based on the boundedness of operators $\sigma_z(w)$ and $\sigma_\tau(\mathbf{u}_\tau)$, as operators from $H_{1/2}^\circ(E^*)$ into $H_{-1/2}(E^*)$. The coercivity of quadratic part of $F$ was established in /14/, while for $F_2$ it is proved similarly.

## 4. Qualitative behavior of the solution and of its integral characteristics.

In the analysis of problems considered here, by virtue of independent determination in them of normal components of unknown functions, we can make use of some results obtained in /2/ for the problem of a cut with partly contacting surfaces without friction.

Thus the statement that if along a certain part $\Gamma_1$ of the cut, the boundary $\Gamma$ of the region of the cut $\Omega$ is widened, with the external normal loads $-r_z - \sigma_z^\circ$ not diminished, then the region of contact will not be narrowed, and of stress intensity factor $K_I$ along the remaining part of the contour $\Gamma \setminus \Gamma_1$ is not diminished.

Let us establish a number of new properties of integral characteristics of the solution.

**4.1.** Let us consider the energy characteristics of the solution which depend on displacements and stresses at the cut surfaces. We limit the case to the consideration of the dependence of the friction coefficient on pressure ($g \not\equiv 0$). The sum of quadratic parts of $F_1$ and $F_2$

$$W = \frac{1}{2} \int [\sigma_z(w) w + \sigma_\tau(\mathbf{u}_\tau) \mathbf{u}_\tau] \, dx \, dy$$

represent the elastic energy of the deformed volume with a cut. The sum of linear parts of $F_1$ and $F_2$

$$-A = \int_\Omega [(\sigma_z^\circ - r_z) w + (\sigma_\tau^\circ - \mathbf{r}_\tau) \mathbf{u}_\tau] \, dx \, dy$$

represent the work of external forces with reverse sign. The nonquadratic part of $F_2$ is the work of friction forces of the cut surface taken with the reverse sign

$$L = \int_\Omega [\sigma_\tau(\mathbf{u}_\tau) - \mathbf{r}_\tau + \sigma_\tau^\circ] \mathbf{u}_\tau \, dx \, dy = -\int_\Omega p f(p) |\mathbf{u}_\tau| \, dx \, dy$$

For the introduced energy characteristics the following theorem is valid.

**Theorem 4.1.** Let the friction coefficient $f(p)$ vary proportionally to the parameter $\gamma > 0$. Then the elastic energy $W$ and the sum of work of external forces and friction forces $L + A$ is a monotonically decreasing function $\gamma$ (the friction coefficient for Coulomb law of friction).

**Proof.** First we consider the behavior of the function $W(\gamma)$. Owing to the splitting of the input problem into two, it is sufficient to establish the decrease of the function

$$W^*(\gamma) = W(w = 0) = \int_\Omega \frac{1}{2} \sigma_\tau(\mathbf{u}_\tau) \mathbf{u}_\tau \, dx \, dy$$

This property of $W^*(\gamma)$ follows directly from the more general conclusion on the decrease of the quadratic part of the functional /15/

$$I(\mathbf{v}(\gamma)) = \min_{\mathbf{u}\in U}\left[I(\mathbf{u}) = \frac{1}{2}a(\mathbf{u},\mathbf{u}) + \gamma j(\mathbf{u}) - M(\mathbf{u})\right]$$

as a function of the parameter $\gamma$. In /15/ $U$ is the Hilbert space, $a(\mathbf{u},\mathbf{u})$ is a bilinear continuous symmetric form that satisfies the condition $a(\mathbf{u},\mathbf{u})\,| \geqslant c\,\|\mathbf{u}\|^2$, $M(\mathbf{u})$ is a continuous linear functional, and $j(\mathbf{u})$ is a convex continuous functional that satisfies condition $j(t\mathbf{u}) = |t|\,j(\mathbf{u})$.

The functional $F_2(\mathbf{u}_\tau)$ and $H_{1/2}{}^\circ(\Omega)$ under conditions of Theorem 3.1 (for $g\equiv 0$) satisfies all requirements stated in /15/ for the functional $I(\mathbf{u})$. Thus it is possible to maintain that the decrease of $W^*(\gamma)$, and with it that of $W(\gamma)$ is monotonic.

For functions $w$ and $\mathbf{u}_\tau$ that minimize $F_1$ and $F_2$ and hence satisfy the boundary conditions, the following equality is satisfied:

$$W = \int_\Omega \left[\frac{1}{2}(\mathbf{r}_\tau - \sigma_\tau{}^\circ)\,\mathbf{u}_\tau - \frac{1}{2}pf(p)\,|\mathbf{u}_\tau| + \frac{1}{2}(r_z - \sigma_z{}^\circ)\,w\right]dx\,dy = \frac{1}{2}(L + A)$$

Thus from the decrease of $W(\gamma)$ follows the decrease of $L(\gamma) + A(\gamma)$.

4.2. Let us now consider other integral characteristics of the solution which are defined by the displacement field of $w$ and $\mathbf{u}_\tau$. We introduce the quantities

$$V_z{}^{ij} = \int_\Omega w x^i y^j\,dx\,dy,\quad \mathbf{V}_\tau{}^{ij}(V_x{}^{ij}, V_y{}^{ij}) = \int_\Omega \mathbf{u}_\tau x^i y^j\,dx\,dy$$

which represent the "moments of distribution" of displacements of the cut surface. Thus, for example, $-2V_z{}^{00}$ represents the volume of the gap bounded by the deformed surfaces of the cut. The quantities $V_z{}^{ij}$, $\mathbf{V}_\tau{}^{ij}$ define the elastic field created by the crack at considerable distances from it /16/. Let the external loads be defined by polynomials of the form

$$f_z = -\sigma_z{}^\circ + r_z = \sum_0^N K_z{}^{ij} x^i y^j \tag{4.1}$$

$$\mathbf{f}_\tau = -\boldsymbol{\sigma}_\tau{}^\circ + \mathbf{r}_\tau = \sum_0^N \mathbf{K}_\tau{}^{ij}(K_x{}^{ij}, K_y{}^{ij})\,x^i y^j$$

Theorem 4.2. Each of the quantities $V_{x(y)(z)}^{ij}$ is a monotonically increasing function of the coefficient $K_{x(y)(z)}^{ij}$ (for constant coefficients with other $i, j$ in expansions (4.1)).

Proof. Let $w(f_z{}^1)$, $w(f_z{}^2)$, $\mathbf{u}_\tau(\mathbf{f}_\tau{}^1)$, $\mathbf{u}_\tau(\mathbf{f}_\tau{}^2)$ be the solutions of variational problems (2.1) and (3.1) for loads which satisfy two sets of coefficients $K_{1z}{}^{ij}$ and $K_{2z}{}^{ij}$, also $\mathbf{K}_{1\tau}{}^{ij}$ and $\mathbf{K}_{2\tau}{}^{ij}$. The necessary conditions of minimum of $F_1(f_z{}^1, w)$ and $F_2(f_z{}^2, w)$, when $w = w(f_z{}^1)$ and $w = w(f_z{}^2)$ with allowance for (2.1) and (4.1) have the form

$$\int_\Omega (f_z{}^1 - f_z{}^2)[w(f_z{}^1) - w(f_z{}^2)]\,dxdy = \sum_0^N (K_{1z}{}^{ij} - K_{2z}{}^{ij})(V_{1z}{}^{ij} - V_{2z}{}^{ij}) = \tag{4.2}$$

$$F_1(f_z{}^1, w(f_z{}^2)) - F_1(f_z{}^1, w(f_z{}^1)) + F_1(f_z{}^2, w(f_z{}^1)) - F_1(f_z{}^2, w(f_z{}^2)) \geqslant 0$$

Let us now assume that $K_{1z}{}^{mn} \geqslant K_{2z}{}^{mn}$ for some $i = m, j = n$ while for the remaining $i, j$ we have $K_{1z}{}^{ij} = K_{2z}{}^{ij}$. Then for inequality (4.2) to be satisfied it is necessary that $V_{1z}{}^{mn} \geqslant V_{2z}{}^{mn}$, i.e. the dependents $V_z{}^{ij}(K_z{}^{ij})$ are monotonically increasing.

Similarly to (4.2) we obtain the inequality

$$\sum_0^N [(K_{1x}^{ij} - K_{2x}^{ij})(V_{1x}^{ij} - V_{2x}^{ij}) + (K_{1y}^{ij} - K_{2y}^{ij})(V_{1y}^{ij} - V_{2y}^{ij})] = \tag{4.3}$$

$$\int_\Omega (\mathbf{f}_\tau{}^1 - \mathbf{f}_\tau{}^2)[\mathbf{u}_\tau(\mathbf{f}_\tau{}^1) - \mathbf{u}_\tau(\mathbf{f}_\tau{}^2)]\,dxdy = F_2(\mathbf{f}_\tau{}^1, \mathbf{u}_\tau(\mathbf{f}_\tau{}^2)) -$$

$$F_2(\mathbf{f}_\tau{}^1, \mathbf{u}_\tau(\mathbf{f}_\tau{}^1)) + F_2(\mathbf{f}_\tau{}^2, \mathbf{u}_\tau(\mathbf{f}_\tau{}^1)) - F_2(\mathbf{f}_\tau{}^2, \mathbf{u}_\tau(\mathbf{f}_\tau{}^2)) \geqslant 0$$

Note that inequality (4.3) is valid when the coefficients $K_z{}^{ij}$ and, consequently, the functions $p$ are the same for the two comparable states. From inequality (4.3) follows the monotonic character of increase of functions $V_{x(y)}^{ij}(K_{x(y)}^{ij})$.

Let us continue the investigation of moments $V_z{}^{ij}$, and establish some of its extremal properties. We consider the set of mixed problems with known boundaries of the contact regions $E^*$ under conditions

$$w(E^*) = 0; \quad w(E^*) \leqslant 0, \quad \sigma_z(w(E^*)) = r_z - \sigma_z^\circ \tag{4.4}$$

As seen from the solution of the input problem (1.6), the $w(E)$ belongs to the set $w(E^*)$ and is distinguished by that the additional inequality in (1.1) is satisfied in it for $\sigma_z(w)$ inside $E$. We shall show that in the solution of problem (1.6) the maximum value of some linear combination of $V_z^{ij}$ is realized.

**Theorem 4.3.** The input problem (1.6) is uniquely characterized by the condition

$$\sum_0^N K_z^{ij} V_z^{ij}(w(E)) = \max_{w(E^*)} \sum_0^N K_z^{ij} V_z^{ij}(w(E^*)) \tag{4.5}$$

**Proof.** Functions $w(E^*)$ are admissible for the variational problem (2.1), hence

$$F_1(w(E)) \leqslant F_1(w(E^*)), \quad \forall w(E^*)$$

But with (2.1) and (4.1) taken into account, we have

$$F_1(w(E^*)) = \int_\Omega \left[\frac{1}{2}\sigma_z(w(E^*))\, w(E^*) - r_z w(E^*) + \sigma_z^\circ w(E^*)\right] dx dy =$$

$$-\int_\Omega \frac{1}{2}(r_z + \sigma_z^\circ)\, w(E^*)\, dx dy = -\frac{1}{2}\sum_0^N K_z^{ij} V_z^{ij}(w(E^*))$$

Similarly

$$F(w(E)) = -\sum_0^N K_z^{ij} V_z^{ij}(w(E))$$

Consequently, for the solution of the input problem condition (4.5) is satisfied. If, however, now condition (4.5) is satisfied for $w(E^*)$, then it cannot be satisfied for other $w(E)$, since owing to the strict convexity, the functional cannot have the same values on various functions.

4.3. The solution of the problem considered here, as was shown above, is uniquely determined by the external load (by coefficients $K_z^{ij}$, $\mathbf{K}_\tau^{ij}$ in expansions (4.1)). Then, also, the moments of solution of any order, in particular higher than $N$ are uniquely determined. We may change the initial statement, and specify some set of moments of solution, when the respective loading and distribution of displacements are determined unambigously. The exact formulation of this feature is provided by the following theorem.

**Theorem 4.4.** Specifying the moments $V_z^{ij}$, $i + j = 0, \ldots, N$ uniquely determines the coefficients $K_z^{ij}$, $i + j = 0, \ldots, N$ in the expansion of loading and the solution of variational problem (2.1) (if it exists for these values of $V_z^{ij}$). Specifying moments $V_\tau^{ij}$, $i + j = 0, \ldots, N$ uniquely determines the coefficients $\mathbf{K}_\tau^{ij}$, $i + j = 0, \ldots, N$ in expansion (4.1) and the solution of variational problem (3.1) (if it exists for these values).

**Proof.** Suppose that $V_{1z}^{ij} = V_{2z}^{ij}$, $\forall i, j$; $0 \leqslant i + j \leqslant N$ and that then functions $f_z^1$, $w(f_z^1)$ and $f_z^2$, $w(f_z^2)$ are different.

The functional $F_1$ is strictly convex, hence its increments in the left-hand side of inequality (4.2) are strictly positive. Thus the assumption about the difference of $w(f_z^1)$ and $w(f_z^2)$ contradicts the equality of moments $V_{1z}^{ij}\, V_{2z}^{ij}$. Consequently the set of quantities $V_z^{ij}$ uniquely determines the solution of (2.1) for $w$. The proof of uniqueness of determination of $u_\tau$ is similarly proved by the set of vectors $V_\tau^{ij}$.

**Remarks.** 1°. For the existence of solution of the problem for given moments of displacement distribution it is necessary that the additional constraints imposed in that case were compatible with input constraints in the form of inequalities.

2°. Solution of the variational problem (2.1), (3.1), as shown by the Theorem 4.4, have a definite symmetry with respect to the set of constants $V_z^{ij}$, $K_\tau^{ij}$ $(\mathbf{V}_z^{ij}, \mathbf{K}_\tau^{ij})$. The part of the functional $F_1(F_2)$ linear in $w$ $(\mathbf{u}_\tau)$ is a bilinear symmetric form with respect to constants $V_z^{ij}$ and $K_z^{ij}$ $(V_\tau^{ij}, K_\tau^{ij})$. If problem (2.1), (3.1) is considered with additional constraints, determined by specifying the moments of solution, then the constants $K_z^{ij}$ $(\mathbf{K}_\tau^{ij})$ coincides with the Lagrange multipliers corresponding to these constraints.

Let us demonstrate this in the case of problem (2.1), when the contact region is absent. Let there be the variational problem

$$\min_{w\leqslant 0} F_1(w) = \int_{\Omega} \left[\frac{1}{2}\sigma_z(w)\,w - \sum_0^N K_z^{ij} V_z^{ij}(w)\right] dx\,dy$$

(where the constants $K^{ij}$ are unknown) with conditions

$$\int_{\Omega} wx^i y^j dxdy = V_z^{ij}, \quad 0 \leqslant i+j \leqslant N$$

Introducing the Lagrange multipliers $\lambda^{ij}$ and varying in the Lagrange functional the function $w$, taking into account the absence of contact region, for the solution of problem in $w^\circ$ we obtain

$$\sigma_z(w^\circ) = r_z - \sigma_z^\circ = \sum_0^N \lambda^{ij} x^i y^j$$

i.e. $K^{ij} = \lambda^{ij}$.

A similar symmetry exists in problems on stresses, for example, in the problem of contact of two elastic bodies /17/. In them the sets of quantities $V_z^{ij}$, and $K_z^{ij}$ ($V_\tau^{ij}$ and $K_\tau^{ij}$) change places. Here $V^{ij}$ are the coefficients specified in the input problem, and $K^{ij}$ are moments of the sought stress distribution.

3°. The Theorems 4.2 – 4.4 are valid not only for polynomial form of external loads, but also for their expansion in any system of functions belonging to $H_{-1/2}(\Omega)$.

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